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The γ-open Open Topology for Function Spaces

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Abstract. In this paper we have introduced the notion of γ- open open topology and proved some properties which the topology does possess. We have also introduced the concept of convergence of nets in $\gamma H(X)$ (where $\gamma H(X)$ is the set of all self γ- homeomorphisms on a topological space X) and showed when $\gamma H(X)$ is complete.

1. Introduction

A set-set topology is one which is defined as follows : Let (X,τ) and (Y,τ^*) be two topological spaces. Let $\mathcal{U}$ and $\mathcal{V}$ be collections of subsets of X and Y respectively. Let $F \subset Y^X$ be a collection of functions from X into Y. We define, for $U \in \mathcal{U}$ and $V \in \mathcal{V}$, $(U,V) = \{f \in F : f(U) \subset V\}$. Let $S(U,V) = \{(U,V) : U \in \mathcal{U}, V \in \mathcal{V}\}$. If $S(U,V)$ is a subbasis for a topology $\tau(U,V)$ on F, then $\tau(U,V)$ is called a set-set topology.

The most commonly discussed set-set topologies are the compact-open topology, τ_{co}, which was introduced in 1945 by R.Fox [4] and the point-open topology, τ_p. For τ_{co}, $\mathcal{U}$ is the collection of all compact subsets of X and $\mathcal{V}$ the collection of all open subsets of Y, while for τ_p, $\mathcal{U}$ is the collection of all singletons in X and $\mathcal{V}$ the collection of all open subsets of Y.

In section 2 of this paper, we shall introduce and discuss the γ open-open topology for function spaces. We shall also show which of the desirable properties $\tau_{\gamma oo}$ possesses. In section 3, we shall introduce the notion of convergence of nets in $(\gamma H(X), \tau_{\gamma oo})$ (where $\gamma H(X)$ is the collection of all self γ -homeomorphisms on X) and the completeness of $\gamma H(X)$.

Throughout this paper, (X,τ) (simply X) and (Y,τ^*) always mean topological spaces. Let S be a subset of X. The closure (resp. interior) of S will be denoted by $cl(S)$ (resp. $int(S)$).

A subset S of X is called a semi-open set [7] if $S \subseteq cl(int(S))$. The complement of a semi-open set is called a semi-closed set. The family of all semi-open sets in a topological space (X,τ) will be denoted by $SO(X)$. A subset $M(x)$ of a space X is called a semi-neighborhood of a point $x \in X$

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if there exists a semi-open set S such that $x \in S \subseteq M(X)$. In [6] Latif introduced the notion of semi-convergence of filters. Let $S(x) = \{A \in SO(X) : x \in A\}$ and let $S_x = \{A \subseteq X :$ there exists $\mu \subseteq S(x)$ such that μ is finite and $\cap\mu \subseteq A\}$.Then S_x is called the semi-neighborhood filter at x. For any filter Γ on X we say that Γ semi-converges to x if and only if Γ is finer than the semi-neighborhood filter at x.

Definition 1.1 ([5])**.** A subset U of X is called a γ open set if whenever a filter Γ semi-converges to x and $x \in U$, $U \in \Gamma$. The complement of a γ-open set is called a γ- closed set.

The intersection of all γ- closed sets containing A is called the γ- closure of A, denoted by $cl_\gamma(A)$.A subset A is γ- closed iff $A = cl_\gamma(A)$. We denote the family of all γ- open sets of (X, τ) by τ^γ. It is shown in [8] that τ^γ is a topology on X. In a topological space (X, τ), it is always true that $\tau \subseteq S(X) \subseteq \tau^\gamma$.

Example 1.2. We now give examples of γ- open sets.

Let $X = \{0, 1, 2, 3\}$,
$\tau = \{\phi, \{0\}, \{1\}, \{0, 1\}, \{0, 2\}, \{0, 1, 2\}, X\}$.
Now, $\{0, 2\}$ and $\{0, 3\}$ are semi-open sets and $\{0\}$ is an element of S_0. For any filter Γ on X, if Γ semi-converges to 0, since Γ includes S_0, then $\{0\}$ is a γ- open set.Also $\{3\}$ is a γ- open set which is not open in X.

Remark 1.3. Every open set of a topological space X is a γ- open set but the converse may not be true.

Definition 1.4 ([8])**.** A function $f : X \to Y$ is γ- continuous if the inverse image of every open set of Y is γ- open in X.

The set of all γ- continuous functions from X into Y is denoted by $\gamma C(X, Y)$.

Definition 1.5 ([8])**.** A function $f : X \to Y$ is said to be γ- irresolute if the inverse image of every γ- open set of Y is γ- open in X.

Definition 1.6. A function $f : X \to Y$ is said to be γ-homeomorphism if it is a bijection so that the image and the inverse image of γ- open sets are γ-open.

The collection of all γ- homeomorphisms from X into Y is denoted by $\gamma H(X, Y)$.

Definition 1.7 ([5])**.** A point $x \in X$ is said to be a γ- interior point of A if there exists a γ- open set U containing x such that $U \subseteq A$.

The set of all γ- interior points of A is said to be γ- interior of A and is denoted by $int_\gamma(A)$.

Theorem 1.8 ([5])**.** *For a subset A of a space X, $int_\gamma(X \setminus A)$=$X \setminus cl_\gamma(A)$.*

2. The γ-open Open Topology

Let $\mathcal{U}$ be the collection of all γ- open sets in X and $\mathcal{V}$ be the collection of all open sets in Y, then $S_{\gamma OO}$=$S(U,V)$ where $U \in \mathcal{U}$ and $V \in \mathcal{V}$ is the subbasis for a topology, $\tau_{\gamma oo}$, on any $F \subset Y^X$, which is called the γ- open open topology.

We now examine some of the properties of function spaces the γ- open open topology possesses.

Theorem 2.1. *Let $F \subset Y^X$. If (Y, τ^*) is T_i, for $i = 0, 1, 2$; then $(F, \tau_{\gamma oo})$ is T_i, for $i = 0, 1, 2$.*

Proof. We shall show the case $i = 2$, the other cases are done similarly. Let $i = 2$. Let $f, g \in F$ be such that $f \neq g$. Then there exists some $x \in X$ such that $f(x) \neq g(x)$. If Y is T_2, then there exists disjoint open sets U and V in Y such that $f(x) \in U$ and $g(x) \in V$. Both f and g are γ- continuous, so there are γ- open sets M and N in X with $x \in M \cap N$, $f(M) \subset U$ and $g(N) \subset V$. Hence, $f \in (M,U)$, $g \in (N,V)$ and $(M,U) \cap (N,V)$=ϕ. Thus $(F, \tau_{\gamma oo})$ is T_2. □

A topology τ^* on $F \subset Y^X$ is called an admissible [1] topology for F provided the evaluation map $E : (F, \tau^*) \times (X, \tau) \rightarrow (Y, \tau')$ defined by $E(f, x) = f(x)$ is continuous.

Theorem 2.2. *If $F \subset C(X,Y)$, then $\tau_{\gamma oo}$ is admissible for F.*

Proof. Let $F \subset C(X,Y)$. Let $V \in \tau'$ and $(f,p) \in E^{-1}(V)$. Then $f(p) \in V$. Since f is continuous, there exists some $U \in \tau$ such that $p \in U$ and $f(U) \subset V$. So $(f,p) \in (U,V) \times U$. Since every open set is a γ open set, U is a γ -open set as well as an open set. If $(g,y) \in (U,V) \times U$, then $g(U) \subset V$ and $y \in U$. So $g(y) \in V$. Hence $(U,V) \times U \subset E^{-1}(V)$. Therefore $\tau_{\gamma oo}$ is admissible for F. □

Remark 2.3. The sets of the form (U,V) where both U and V are γ-open sets in X form a subbasis for $(\gamma H(X), \tau_{\gamma oo})$.

Let $(G, \circ)$ be a group such that (G,T) is a topological space, then (G,T) is a topological group provided the two maps are continuous 1) $m : G \times G \rightarrow G$ is defined by $m(g_1, g_2) = g_1 \circ g_2$ and 2) $\Phi : G \rightarrow G$ defined by $\Phi(g) = g^{-1}$. If only the first map is continuous, then we call (G,T) a quasi-topological group [9].

Note that $\gamma H(X)$ with the binary operation $\circ$, compositions of functions, and identity element e, is a group.

Theorem 2.4. *Let X be a topological space and let G be a subgroup of $\gamma H(X)$. Then $(G, \tau_{\gamma oo})$ is a topological group.*

Proof. Let X be a topological space and G be a subgroup of $\gamma H(X)$. We have to prove that the two maps $m : G \times G \rightarrow G$ defined by $m(g_1, g_2) = g_1 \circ g_2$ and $\Phi : G \rightarrow G$ defined by $\Phi(g) = g^{-1}$ are continuous.

Let (U, V) be a subbasic open set in $\tau_{\gamma oo}$ such that both U and V are γ-open sets. Let $(f, g) \in m^{-1}((U, V))$. Then $f \circ g(U) \subset V$ and $g(U) \subset f^{-1}(V)$. So $(f, g) \in (g(U), V) \times (U, g(U)) \in \tau_{\gamma oo} \times \tau_{\gamma oo}$. But $(g(U), V) \times (U, g(U)) \subset m^{-1}((U, V))$. Thus m is continuous.

Now the inverse map $\Phi : G \to G$ is bijective and $\Phi^{-1} = \Phi$. Thus in order to show that Φ is continuous, it is sufficient to show that Φ is an open map. Let (U, V) be a subbasic open set in $\tau_{\gamma oo}$ where U and V are both γ- open sets. Now $\Phi((U, V)) = ((X \setminus V, X \setminus U))$; since we are dealing with γ- homeomorphisms. Now, if C and D are γ- closed sets, then $int_\gamma C$ and $int_\gamma D$ are γ- open sets (using Theorem 1.8). Thus, since $(X \setminus V), (X \setminus U)$ are γ- closed sets, $int_\gamma(X \setminus V), int_\gamma(X \setminus U)$ are γ- open sets. Again since G is a set of γ- homeomorphisms, $(X \setminus V, X \setminus U) = (int_\gamma(X \setminus V), int_\gamma(X \setminus U))$ but this is in $\tau_{\gamma oo}$. Therefore $\Phi(U, V)$ is an open set in $\tau_{\gamma oo}$. So, Φ is open and our theorem is proved. □

3. COMPLETENESS OF $(\gamma H(X), \tau_{\gamma oo})$

We now introduce the notion of convergence of nets in $(\gamma H(X), \tau_{\gamma oo})$ and the completeness of $(\gamma H(X), \tau_{\gamma oo})$. For this purpose, we require the following definitions and theorems.

Definition 3.1. A net in a set X (where X is a topological space) is a map $x : \Lambda \to X$ (Λ is a directed set). We often write such a net by the symbol $\{x_\lambda : \lambda \in \Lambda\}$ writing x_λ instead of $x(\lambda)$.

Definition 3.2. A net $\{x_\lambda : \lambda \in \Lambda\}$ in X is said to be converge to a limit $x \in X$ (in symbol $x_\lambda \to x$) if for every neighborhood V of x, $\exists$ a $\lambda_0 \in \Lambda$ such that $\lambda \geq \lambda_0$ implies $x_\lambda \in V$.

Definition 3.3. A net $\{x_\lambda : \lambda \in \Lambda\}$ in X is said to γ- converge to a limit $x \in X$ (in symbol $x_\lambda \to^\gamma x$) if for every γ- open set V containing x, $\exists\ \lambda_0 \in \Lambda$ such that $\lambda \geq \lambda_0$ implies $x_\lambda \in V$. We often denote this by $\gamma \lim_\lambda x_\lambda = x$.

Theorem 3.4. *A function $f : X \to Y$ (where X and Y are topological spaces) is γ- irresolute at a point $x \in X$ iff for any net $\{x_\lambda : \lambda \in \Lambda\}$ in X γ- converging to x, the net $\{f(x_\lambda) : \lambda \in \Lambda\}$ γ- converges to $f(x)$ in Y.*

Proof. First assume that f is γ- irresolute at $x \in X$. Let $\{x_\lambda : \lambda \in \Lambda\}$ be a net in X γ- converging to x. Let V be a γ- open set in Y containing $f(x)$. Now $\exists$ a γ- open set U containing x in X such that $f(U) \subset V$ Now $\{x_\lambda : \lambda \in \Lambda\}$ γ- converges to x implies $\exists\ \lambda_0 \in \Lambda$ such that $x_\lambda \in U$, $\forall \lambda \geq \lambda_0$. Hence, $\forall \lambda \geq \lambda_0$, $f(x_\lambda) \in V$. This shows that $\{f(x_\lambda) : \lambda \in \Lambda\}$ lies eventually in V and hence it γ- converges to $f(x)$.

To prove the converse, assume that f is not γ- irresolute at x. Then $\exists$ a γ- open set W containing $f(x)$ in Y such that from every γ- open set U containing $x \in X$, $\exists$ an element x_U with $f(x_U) \notin W$. Let $\gamma\mathcal{N}_x$ be the γ-neighborhood system at x. So, $\{x_U : U \in \gamma\mathcal{N}_x\}$ is a net in X γ- converging to x, but the net $\{f(x_U) : U \in \gamma\mathcal{N}_x\}$ in Y does not lie eventually in W and consequently it cannot γ- converge to $f(x)$. □

Theorem 3.5. *Let $\{h_\nu : \nu \in \mathcal{V}\}$ be a net in the group $\gamma H(X)$ of self γ- homeomorphisms of a topological space X. Then $h_\nu \to h$ in $\tau_{\gamma oo}$ iff $h_\nu(x_\delta) \to^\gamma h(x)$ whenever $x_\delta \to^\gamma x$ in X.*

Proof. First assume that, $h_\nu \to h$ in $\tau_{\gamma oo}$. Let (U, V) (U, V both are γ- open sets of X)be an open set in $(\gamma H(X), \tau_{\gamma oo})$ containing h. Then $\exists\ \nu_0 \in \mathcal{V}$ such that $h_\nu \in (U, V)$, $\forall \nu \geq \nu_0$ ie, $h_\nu(U) \subset V$, $\forall \nu \geq \nu_0$.Now, let $x_\delta \to^\gamma x$ in X. Then for every γ- open set U containing x, $\exists\ \delta_0 \in D$ such that $x_\delta \in U$, $\forall \delta \geq \delta_0$.Hence, $\forall\ \nu \geq \nu_0, \delta \geq \delta_0$; $h_\nu(x_\delta) \in V$. Also, $h(x) \in V$. Hence $\{h_\nu(x_\delta) : \nu \in \mathcal{V}, \delta \in D\}$ γ- converges to $h(x)$ whenever $x_\delta \to^\gamma x$.

Next, if possible, let $h_\nu \not\to h$ in $\tau_{\gamma oo}$. Then $\exists$ a neighborhood (U, V) (U, V both are γ- open sets of X)containing h such that $\forall\ \nu \in \mathcal{V}$, $h_\nu \notin (U, V)$ ie, $h_\nu(U) \not\subset V$. So from every γ- open set U containing x, $\exists$ an element x_U with $h_\nu(x_U) \notin V$. Let $\gamma\mathcal{N}_x$ be the γ- neighborhood system at x. Now $h \in (U, V)$ implies $h(U) \subset V$. Hence, $\forall\ \nu \in \mathcal{V}$, $U \in \gamma\mathcal{N}_x$, $h_\nu(x_U) \notin h(U)$, ie, $h_\nu(x_U) \not\to^\gamma h(x)$. Contrapositively, we can say that whenever $h_\nu(x_U) \to^\gamma h(x)$ for $x_U \to^\gamma x$; $h_\nu \to h$ in $\tau_{\gamma oo}$ □

Now we define a uniformity $\mathcal{U}_o$ on $\gamma H(X)$ by defining $(x, y) \in U_o$ if $xy^{-1} \in U$ and $yx^{-1} \in U$ where U is a neighborhood of the identity in $\gamma H(X)$. Then $(\gamma H(X), \mathcal{U}_o)$ becomes a uniform space.

Definition 3.6. A net $\{h_\nu : \nu \in \mathcal{V}\}$ in $(\gamma H(X), \mathcal{U}_o)$ is called a Cauchy net if for each $U \in \mathcal{U}_o$, $\exists$ a $\nu_0 \in \mathcal{V}$ such that $\nu_1, \nu_2 > \nu_0$ implies $(h_{\nu_1}, h_{\nu_2}) \in U$. If every Cauchy net in $\gamma H(X)$ converges (has a limit in $\gamma H(X)$), then $\gamma H(X)$ will be called complete (in the structure $\mathcal{U}_o$).

Definition 3.7. A topological space X is said to be γ-regular if for each open set U of X and each $x \in U$, there exists a γ- open set V in X, such that $x \in V \subseteq U$.

Example 3.8. We give an example of a γ- regular space which is not regular.Consider
$X = \{0, 1, 2, 3, 4, 5\}$
$\tau = \{\phi, \{0, 2, 4\}, \{3, 5\}, \{0, 2, 3, 4, 5\}, \{2, 4\}, \{2, 3, 4, 5\}, X\}$
Now $\{1\}$ is a closed set and $0 \notin \{1\}$. But 0 and $\{1\}$ cannot be strongly separated. Hence, X is not regular. Now, $\{3, 5\}$, $\{0, 2, 4\}$, $\{1, 2, 4\}$, $\{1, 3, 5\}$, $\{0, 2, 3, 4, 5\}$ are semi-open sets. Also, $\{3, 5\} \in S_3$, $\{3, 5\} \in S_5$, $\{2, 4\} \in S_2$, $\{2, 4\} \in S_4$, $\{0, 2, 4\} \in S_0$, $\{1\} \in S_1$. For any open neighborhood U_0 of 0, $0 \in \{0, 2, 4\} \subseteq U_0$; for any open neighborhood U_1 of 1, $1 \in \{1\} \subset U_1$; for any open neighborhood U_2 of 2, $2 \in \{2, 4\} \subseteq U_2$; for any open neighborhood U_3 of 3, $3 \in \{3, 5\} \subseteq U_3$; for any open neighborhood U_4 of 4, $4 \in \{2, 4\} \subseteq U_4$; for any open neighborhood U_5 of 5, $5 \in \{3, 5\} \subseteq U_5$. Hence X is γ- regular.

Theorem 3.9. *$(\gamma H(X), \tau_{\gamma oo})$ is complete if X is γ- regular and complete.*

Proof. Let $\{h_\nu : \nu \in \mathcal{V}\}$ be a Cauchy net in $\gamma H(X)$ (relative to $\mathcal{U}_o$) ie, $h_\nu {h_\mu}^{-1} \to$ identity and $h_\mu {h_\nu}^{-1} \to$ identity, for $\mu, \nu \in \mathcal{V}$.Also, for each $x \in X$, $\{h_\nu(x) : \nu \in \mathcal{V}\}$ is a Cauchy net in X and hence converges for each $x \in X$. Let its limit be $h(x)$. We will show that whenever a net $\{x_\delta : \delta \in D\}$ in X γ- converges to $x \in X$, the net $h_\nu(x_\delta) \to^\gamma h(x)$.If possible, let $h_\nu(x_\delta) \not\to^\gamma h(x)$ and suppose that $h_\nu(x_\delta) \to^\gamma y \neq h(x)$. Since X is γ- regular, $h_\nu(x_\delta) \to y \neq h(x)$. Now, $\lim\limits_{\nu,\delta} h_\nu(x_\delta) = y$, $\lim\limits_{\delta} h_\nu(x_\delta) = h_\nu(x) \to h(x)$. Since, $\lim\limits_{\nu,\delta} h_\nu(x_\delta) = \lim\limits_{\nu} \lim\limits_{\delta} h_\nu(x_\delta)$, therefore $y = h(x)$ and this contradiction proves that $h_\nu(x_\delta) \to^\gamma h(x)$. Thus $h_\nu \to h$ in $\tau_{\gamma oo}$. We now show that h is γ- irresolute at the point $x \in X$. We know that if $x_\delta \to^\gamma x$, then $h_\nu(x_\delta) \to^\gamma h(x)$. Therefore, $\gamma\lim\limits_{\nu,\delta} h_\nu(x_\delta) = \gamma\lim\limits_{\nu} \gamma\lim\limits_{\delta} h_\nu(x_\delta) = \gamma\lim\limits_{\delta} h(x_\delta)$ and hence $h(x_\delta) \to^\gamma h(x)$. Thus we have, whenever $x_\delta \to^\gamma x$, $h(x_\delta) \to^\gamma h(x)$. Using Theorem 3.4 we can show that h is γ- irresolute at $x \in X$. Since the conditions on h are equivalent to the same conditions on h^{-1}, we have $h \in \gamma H(X)$ and hence $(\gamma H(X), \tau_{\gamma oo})$ is complete. □

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